

unique in the class of analytic functions we have $u_2(r) = k^2 u_1(kr)$ for $r < 1$. Setting in this identity the limiting value $r = 1$, we find that $u_2(1) = k^2 u_1(k) < 0$, which contradicts the boundary condition for the solution u_2 .

We have thus established that the nontrivial solution of the boundary-value problem (6.1) in the circle $\Omega \subset R^2$ is unique.

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CONSERVATION LAWS, INVARIANCE, AND THE EQUATIONS OF GAS DYNAMICS

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In a large number of papers by L. V. Ovsyannikov, his students, and followers, an analysis was made of group properties of many equations of mathematical physics and it was shown that a knowledge of group properties of the equations is useful for their classification and for obtaining particular solutions (see, for example, [1-3]). An inverse formulation of the problem is also possible: from a given group, sometimes with an additional assumption concerning the transformation law for the desired quantities, to seek the class of differential equations invariant with respect to this group [4, 5]. A similar problem arose, in effect, at roughly the same time, from the theory of relativity, wherein the physics and mathematics began to search for equations describing the dynamics of some range of phenomena dependent on a knowledge of the laws of invariance. From this standpoint the most fundamental object turns out to be a group, and the dynamic equation, in its way, turns out to be the "differential representation" of this group. And just as there exists a supporting theory of linear representations of groups, there exists, indeed, a theory of "differential representations." In this regard, it is necessary to turn our attention to the importance of rational structural limitations of the class of equations being sought. The fundamental equations of mechanics and theoretical physics possess a definite structure. They are usually quasilinear and admit a complete set of conservation laws (see Sec. 1), and, consequently, have a symmetric structure [6-12]. A second simple, but useful observation, consists in the fact that the quantities being sought have, as a rule, a specific tensor type (scalars, vectors, etc.) with respect to suitable transformations. This holds even for the basic conservation laws (mass, momentum, et al.). Only in quantum mechanics do quantities of another

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kind appear, namely, spinors. By the same token, both the quantities being sought as well as the basic conservation laws may be classified beforehand corresponding to the simplest tensor types (more generally, in accordance with irreducible representations of fundamental groups). Furthermore, invariant equations may be found starting from this classification. Hence, at least in principle, a complete classification of elementary types of differential equations of mechanics and theoretical physics may be obtained, i.e., a complete structural enumeration of their possible forms (to within empirically determined constants and equations of state). For the description of "component" types ("mixtures," etc.) additional structural constraints may apparently be necessary. From the logical point of view, a similar classification must lie at the foundation of a general theory of mathematical physics.

In the present paper we illustrate some characteristic aspects of the problem by way of the equations of gas dynamics, both Galilean and Lorentzian.

1. Conservation Laws. Let $x \equiv (x^0, \dots, x^n) \in \Omega$, where Ω is a domain in R^{n+1} , $w(x) \in R^m$, $\varphi^k(w(x)) \in R^S$ ($k = 0, \dots, n$), $f(w(x)) \in R^S$.

In connection with the basic aims of this paper we consider conservation laws of only the first order with respect to derivatives and with functions appearing in them depending explicitly only on w .

By a conservation law we mean a relationship of the form

$$\sum_{k=0}^n \frac{\partial \varphi^k}{\partial x^k} = f, \quad (1.1)$$

satisfied throughout Ω . A conservation law is exact if $f = 0$.

Usually, the functions $w(x)$ satisfy some condition (belonging to class K), for example, some equation. In this case, it is required that Eq. (1.1) hold for arbitrary $w \in K$, for example for an arbitrary smooth solution. We denote the Euclidean scalar product of vectors a and b by $(a|b)$.

A system of conservation laws (1.1) is called complete if

a) the number of equations is equal to the dimension of w , i.e., $s = m$;

b) there exists a smooth function $u(w) \equiv (u_{(1)}, \dots, u_{(m)})$ such that after scalar multiplication of Eq. (1.1) by u we obtain yet another conservation law

$$\sum_{k=0}^n \frac{\partial \psi^k}{\partial x^k} = h, \quad (1.2)$$

where $\psi^k = \psi^k(w) \in R$; $h = h(w) \in R$;

$$d\psi^k \equiv (u|d\varphi^k), \quad h = (u|f); \quad (1.3)$$

c) the mapping $w \rightarrow u(w)$ is locally one-to-one.

We speak of the conservation laws (1.1) as basis laws and of the conservation law (1.2) as a closing law; the functions $u_{(v)}$ are integrating factors.

By virtue of c) we assume that $w = w(u)$, while φ^k , ψ , f , and h are considered as functions of the integrating factor u (locally). In what follows, everything will be done locally, where such inversion is possible. For simplicity we assume that $f = 0$.

2. Galilean Systems (see also [11, 13, 14]). Throughout this section repeated Latin indices will indicate summation from 1 to 3. Let us assume we have a physical system S for which the invariance principle with respect to the Galilean group Γ holds, including orthogonal transformations of coordinates (x^1, x^2, x^3) , forming the group $SO(3)$, and the Galilean transformations

$$x^0 \rightarrow x^0, \quad x^k \rightarrow x^k + L^k x^0 \quad (k = 1, 2, 3; L^k = \text{const}) \quad (2.1)$$

(x^0 is the time coordinate). We assume that state S can characterize the set of quantities $v(x) \equiv (v^1, v^2, v^3)$ and $q(x) \equiv (q_{(1)}, \dots, q_{(v)})$, so that $w = (v, q)$, $m = 3 + v$. We as-

sume that v has the meaning of velocity, i.e., with respect to transformations of the group $SO(3)$, v^k are transformed as a vector and, with respect to the transformations (2.1), are transformed according to the rule $v^k \rightarrow v^k + U^k$. We assume that the quantities $q(\alpha)$, with respect to all transformations of Γ , are transformed as scalars $q(\alpha) \rightarrow q(\alpha)$. If we introduce the Galilean 4-vector of velocity $V \equiv (V^0, V^1, V^2, V^3)$, putting $V^0 = 1$, $V^k = v^k$ ($k = 1, 2, 3$), it turns out, with respect to all transformations of Γ , that V transforms as a contravariant vector. Thus, by hypothesis, state S is characterized completely by a single vector field $V(x)$ of a special form ($V^0 = 1$) and by the scalar fields ($q(\alpha)$).

We assume that a change of state S is described by the system

$$\frac{\partial w}{\partial x^0} + A^k(w) \frac{\partial w}{\partial x^k} = 0, \quad (2.2)$$

where A^k are matrices. In order that system (2.2) be invariant with respect to the indicated transformations of group Γ , it is necessary and sufficient that Eq. (2.2) be written as

$$\frac{\partial v}{\partial x^0} + v^k \frac{\partial v}{\partial x^k} + \lambda \operatorname{rot} v + \sum_{\alpha} \eta^{(\alpha)} \operatorname{grad} q(\alpha) = 0; \quad (2.3a)$$

$$\frac{\partial \eta}{\partial x^0} + v^k \frac{\partial \eta}{\partial x^k} + \sigma \operatorname{div} v = 0 \quad (2.3b)$$

$$(\lambda = \lambda(q), \eta^{(\alpha)} = \eta^{(\alpha)}(q), \sigma = \sigma(q)).$$

Assume for definiteness that $v \geq 3$. In order that system (2.3) admit conservation laws, functions λ , $\eta^{(\alpha)}$, and σ must satisfy conditions more conveniently formulated in terms of other functions. We are interested only in equations admitting a complete set of conservation laws. For this it is necessary, as a consequence of Eqs. (2.3), that the conservation laws be:

$$\frac{\partial \rho}{\partial x^0} + \frac{\partial (\rho v^k)}{\partial x^k} = 0; \quad (2.4)$$

$$\frac{\partial (\rho v^i)}{\partial x^0} + \frac{\partial}{\partial x^k} (\rho v^i v^k + \delta^{ik} P) = 0 \quad (i = 1, 2, 3); \quad (2.5)$$

$$\frac{\partial}{\partial x^0} \left(E + \rho \frac{|v|^2}{2} \right) + \frac{\partial}{\partial x^k} \left[v^k \left(E + P + \rho \frac{|v|^2}{2} \right) \right] = 0; \quad (2.6)$$

$$\frac{\partial \varphi^{(\alpha)}}{\partial x^0} + \frac{\partial (\eta^{(\alpha)} v^k)}{\partial x^k} = 0 \quad (\alpha = 1, \dots, v-1), \quad (2.7)$$

where $\rho = \rho(q)$, $P = P(q)$, $E = E(q)$, $\varphi^{(\alpha)} = \varphi^{(\alpha)}(q)$, and δ^{ik} is the Kronecker symbol. Functions ρ , P , E , and $\varphi^{(\alpha)}$ are connected by relations which may be described with the aid of a differential form, described below. We recall that all constructions are made locally, so that the transition from Eqs. (2.3) to Eqs. (2.4)-(2.7) is possible, generally speaking, only locally in a suitable image of a chosen neighborhood of some point q_0 . The choice of this neighborhood is determined by conditions from which it follows, in particular, that $\rho \neq 0$. In order to be able to write a condition for a relationship of ρ , P , E , and $\varphi^{(\alpha)}$ in a suitable form, we need to determine which of the conservation laws is to be taken as closing laws and which as basis laws. This is done most clearly in Sec. 3 for Lorentzian equations, where a somewhat different set of hypotheses is adopted. In the meantime, we simply analyze two fundamental possibilities.

A. Assume that a closing conservation law coincides with one of Eqs. (2.7). Changing the notation somewhat and using the fact that $\rho \neq 0$, we write Eqs. (2.4)-(2.7) as

basis conservation laws

$$\frac{\partial \rho}{\partial x^0} + \frac{\partial (\rho v^k)}{\partial x^k} = 0; \quad (2.8)$$

$$\frac{\partial (\rho v^i)}{\partial x^0} + \frac{\partial}{\partial x^k} (\rho v^i v^k + \delta^{ik} P) = 0 \quad (i = 1, 2, 3); \quad (2.9)$$

$$\frac{\partial}{\partial x^0} \left[\rho \left(\varepsilon + \frac{|v|^2}{2} \right) \right] + \frac{\partial}{\partial x^k} \left[\rho v^k \left(\varepsilon + P/\rho + \frac{|v|^2}{2} \right) \right] = 0; \quad (2.10)$$

$$\frac{\partial (\rho c^{(\alpha)})}{\partial x^0} + \frac{\partial (\rho v^k c^{(\alpha)})}{\partial x^k} = 0 \quad (\alpha = 1, \dots, v-2); \quad (2.11)$$

closing conservation law

$$\frac{\partial(\rho s)}{\partial x^0} + \frac{\partial(\rho v^k s)}{\partial x^k} = 0. \quad (2.12)$$

In order for system (2.8)-(2.12) to satisfy a condition of completeness for conservation laws, the function ρ , P , ε , $c^{(\alpha)}$, and s must satisfy the relation (again in a suitable neighborhood)

$$Tds = d\varepsilon + Pd(1/\rho) - \sum_{\alpha=1}^{v-2} \mu_{(\alpha)} dc^{(\alpha)} \quad (2.13)$$

(T and $\mu_{(\alpha)}$ are functions of q). This is the well-known Gibbs relation from thermodynamics. Applying the usual thermodynamic interpretation, we can say that s is the specific entropy, ε is the specific internal energy, ρ is mass density, $c^{(\alpha)}$ is the concentration of some phase, T is the absolute temperature, P is the pressure, and $\mu_{(\alpha)}$ is the chemical potential.

Condition (2.13) is a consequence of the completeness requirement for conservation laws (2.8)-(2.12). Conversely, if we have the equations (2.8)-(2.11), relation (2.13) is satisfied, and $T \neq 0$, the equation (2.12) then follows from Eqs. (2.8)-(2.11). To see this, we need to multiply Eq. (2.8) by $(-z + |v|^2/2)/T$, Eq. (2.9) by $-v^i T$, Eq. (2.10) by $1/T$, and Eq. (2.11) by $-\mu_{(\alpha)}/T$, where $z \equiv \varepsilon + P/\rho - Ts - \sum_{\alpha} \mu_{(\alpha)} c^{(\alpha)}$. After adding and taking relation (2.13) into account, we obtain Eq. (2.12) (z is the specific Gibbs thermodynamic potential).

B. Let Eq. (2.10) be a closing conservation law. A similar case holds, for example, in the theory of long waves (of course, the equations then are not three-dimensional, but one- or two-dimensional) in describing the transport of ingredients (moreover, right-hand terms also appear, but this is not important now). We write Eqs. (2.4)-(2.7) as

basis conservation laws

$$\frac{\partial \rho}{\partial x^0} + \frac{\partial(\rho v^k)}{\partial x^k} = 0; \quad (2.14)$$

$$\frac{\partial(\rho v^i)}{\partial x^0} + \frac{\partial}{\partial x^k} (\rho v^i v^k + \delta^{ik} P) = 0; \quad (2.15)$$

$$\frac{\partial(\rho c^{(\alpha)})}{\partial x^0} + \frac{\partial}{\partial x^k} (\rho v^k c^{(\alpha)}) = 0 \quad (\alpha = 1, \dots, v-1); \quad (2.16)$$

closing conservation law

$$\frac{\partial}{\partial x^0} \left[\rho \left(\varepsilon + \frac{|v|^2}{2} \right) \right] + \frac{\partial}{\partial x^k} \left[\rho v^k \left(\varepsilon + \frac{P}{\rho} + \frac{|v|^2}{2} \right) \right] = 0. \quad (2.17)$$

Functions ρ , P , $c^{(\alpha)}$, and ε must be connected by the relation

$$d\varepsilon = -Pd(1/\rho) + \sum_{\alpha=1}^{v-1} \mu_{(\alpha)} dc^{(\alpha)} \quad (2.18)$$

($\mu_{(\alpha)}$ are functions of q). Equation (2.18) is a consequence of the completeness requirement for conservation laws (2.14)-(2.17). Conversely, assume we have Eqs. (2.14)-(2.16) and that relation (2.18) holds. Multiply Eq. (2.14) by $z - |v|^2/2$, Eq. (2.15) by v^i , and Eq. (2.16) by $\mu_{(\alpha)}$, where $z \equiv \varepsilon + \frac{P}{\rho} - \sum_{\alpha} \mu_{(\alpha)} c^{(\alpha)}$. After summation, with Eq. (2.18) taken into account, we obtain Eq. (2.17).

Thus, the combination of requirements of invariance and conservation law completeness rigidly determines the structure of the equations of gas dynamics. The Gibbs relation (2.13) and its analog (2.18) are also obtained automatically.

Remark 2.1. The approach we have described is based on separating the invariance hypothesis leading to Eqs. (2.3) and the conservation law completeness hypothesis, ensuring the possibility of a passage from Eqs. (2.3) to Eqs. (2.4)-(2.7) by means of the complemented Gibbs relation or its appropriate analog. Such an approach seems reasonable. However, in this connection there also arise negative aspects. Equations (2.3) and (2.4)-(2.7) are

equivalent except, possibly, for certain cases of degeneracy. Nevertheless, however, an additional hypothesis is necessary for the assertion of equivalence. As a result, the domain in which Eqs. (2.3) are reduced to Eqs. (2.4)-(2.7) does not coincide, generally speaking, with the domain of biuniqueness of the mapping $w \rightarrow u(w)$ (it may be a smaller domain). This makes precise formulations more involved and, it seems, not altogether natural. The perception of arbitrariness can furnish a method of separating Eqs. (2.4)-(2.7) into basis relations and closing relations. In Sec. 3 we take a different approach for Lorentzian equations, where from the very beginning the conservation laws are classified according to tensor types. The resulting set of assumptions is, in my view, simpler and more natural. A similar approach is, of course, possible even for Galilean equations [with certain changes, since in some cases one operates with the group $SO(3)$ and in others with the group Γ ; see Remark 2.2]. However, to avoid repetition we restrict our analysis to the Lorentzian equations.

Remark 2.2. In accordance with a tradition of classical mechanics we employ three-dimensional symbolics. A four-dimensional notation is also possible. We put $T^{ij} \equiv \rho V^i V^j + \delta_{(3)}^{ij} P$, where V^i are the components of the Galilean 4-velocity, $i, j = 0, 1, 2, 3$,

$$\delta_{(3)}^{ij} \equiv \begin{cases} 1, & \text{if } i = j, \quad i \neq 0, \\ 0, & \text{if } i \neq j, \text{ or } i = 0, \text{ or } j = 0. \end{cases}$$

It is easy to verify that (T^{ij}) is a contravariant tensor with respect to all transformations from Γ . We write Eqs. (2.4) and (2.5) as

$$\sum_{j=0}^3 \frac{\partial T^{ij}}{\partial x^j} = 0 \quad (i = 0, 1, 2, 3), \quad (2.19)$$

and Eq. (2.7) in the form

$$\sum_{j=0}^3 \frac{\partial (V^j \varphi^{(\alpha)})}{\partial x^j} = 0. \quad (2.20)$$

The left side of Eq. (2.19) is a contravariant vector with respect to all transformations from Γ , while the left side of Eq. (2.20) is a scalar. Equations (2.6) naturally are not written in the four-dimensional tensor symbolics. This is due to the fact that there is no nondegenerate metric for the Galilean group (for the Lorentz group there is the Minkowski metric). We cannot therefore proceed from the contravariant 4-vector (V^j) to a covariant one (the index cannot be omitted). The left side of Eq. (2.6) is a scalar relative to $SO(3)$. Invariance of Eq. (2.6) with respect to Γ is a consequence, for example, of the fact that Eq. (2.6) follows from the invariant equations (2.3). Integrating factors are also classified according to tensor type.

3. Lorentzian Systems. We consider now a physical system S invariant with respect to the Lorentz group L , including transformations of group $SO(3)$ and Lorentz transformations. We assume that state S can be completely characterized by a 4-vector contravariant field $Q = (Q^0, Q^1, Q^2, Q^3)$ and by scalar fields $q(x) \equiv (q_1, \dots, q_\nu)$. For definiteness we assume that $\nu \geq 2$. The quantity

$$\sigma(x) \equiv Q^0 Q^0 - \frac{1}{c^2} \sum_{k=1}^3 Q^k Q^k \quad (3.1)$$

is an invariant of group L (c is the speed of light in a vacuum). We fix point x_0 . Three cases are possible: $\sigma(x_0) > 0$, $\sigma(x_0) = 0$, and $\sigma(x_0) < 0$. In this paper we analyze only the first case. By virtue of the smoothness of $\sigma(x)$ there exists a neighborhood $G(x_0)$ of point x_0 where $\sigma(x) > 0$. We assume further that $x \in G(x_0)$. We put $V^i \equiv Q^i / \sqrt{\sigma}$ ($i = 0, 1, 2, 3$). By virtue of relation (3.1),

$$V^0 V^0 - \frac{1}{c^2} \sum_{k=1}^3 V^k V^k = 1. \quad (3.2)$$

We can now assume that state S is defined by a single vector field $V(x)$, normalized in accordance with relation (3.2), and $(\nu + 1)$ scalar fields $(\sigma, q) \equiv (\sigma, q_1, \dots, q_\nu)$. We call vector V the Lorentzian 4-velocity. Repeated Latin indices will indicate summation from 0 to 3.

We consider the divergence differential expression $J = \partial\phi^j/\partial x^j$, where ϕ^j depends only on (V, σ, q) . In order that J be a scalar (invariant) with respect to all transformations of group L , it is necessary and sufficient that ϕ^j have the form $\phi^j = \varphi(\sigma, q)V^j + \text{const}$.

Consider now the differential expressions $F^i \equiv \partial T^{ij}/\partial x^j$ ($i = 0, 1, 2, 3$). Here T^{ij} depends only on (V, σ, q) . In order that the set of quantities (F^i) transform as a contravariant vector with respect to all transformations from L , it is necessary and sufficient that $T^{ij} = \psi(\sigma, q)V^iV^j + g^{ij}P(\sigma, q) + \text{const}$, where g^{ij} is a contravariant Minkowski tensor [$g^{00} = -1/c^2$, $g^{ii} = 1$ ($i = 1, 2, 3$), $g^{ij} = 0$ ($i \neq j$)]. It is convenient to introduce a quantity ρ by the equation $\psi \equiv \rho + P/c^2$.

In accordance with the program outlined in the introduction, we now classify conservation laws, describing the dynamics of S , according to elementary tensor types. Namely, since we have $3 + 1 + v$ unknowns, we assume that the basis conservation laws include one four-vector relation and v scalar ones, so that the basis laws may be written as

$$\frac{\partial}{\partial x^j} [(\rho + P/c^2)V^iV^j + g^{ij}P] = 0 \quad (i = 0, 1, 2, 3); \quad (3.3)$$

$$\frac{\partial}{\partial x^j} (NV^j) = 0; \quad (3.4)$$

$$\frac{\partial}{\partial x^j} (C^{(\alpha)}V^j) = 0 \quad (\alpha = 1, \dots, v-1) \quad (3.5)$$

[$\rho = \rho(\sigma, q)$, $P = P(\sigma, q)$, $N = N(\sigma, q)$, $C^{(\alpha)} = C^{(\alpha)}(\sigma, q)$]. For simplicity, we assume the conservation laws to be exact. Besides, we can also analyze a more general case. We take the closing conservation law as the scalar:

$$\frac{\partial}{\partial x^j} (SV^{(j)}) = 0 \quad (3.6)$$

[$S = S(\sigma, q)$]. Thus, classification according to elementary tensor types has led to a completely determined form of the equations containing the scalar functions ρ , P , N , $C^{(\alpha)}$, and S (this classification characterizes the fundamental form of gas dynamics), which cannot be assigned arbitrarily; they are connected by a condition, stemming from a requirement as yet not used, relating to the integrating factors. By definition, $\eta_i(V, \sigma, q)$, $\zeta(V, \sigma, q)$, and $\xi_{(\alpha)}(V, \sigma, q)$ exist such that after multiplying Eq. (3.3) by η_i , Eq. (3.4) by ζ , and Eq. (3.5) by $\xi_{(\alpha)}$, and summing, we obtain Eq. (3.6). Moreover, equations analogous to Eq. (1.3) must be satisfied.

We assume, additionally, that (η_i) transform as a covariant vector, ζ and $\xi_{(\alpha)}$ as scalars with respect to all transformations from L . By the same token, factors completely in the spirit of our program may be classified and integrated according to the elementary tensor types. Furthermore, this hypothesis strongly reduces the calculations. But it should be noted that logically it is not necessary and can be weakened. It follows from the hypothesis that the integrating factors must have the structure $\eta_i = \tau(\sigma, q)V_i$, $\zeta = \zeta(\sigma, q)$, and $\xi_{(\alpha)} = \xi_{(\alpha)}(\sigma, q)$ ($V_i \equiv g_{ij}V^j$, g_{ij} is a Minkowski covariant tensor).

According to the definition in Sec. 1, the mapping $(V, \sigma, q) \rightarrow (\eta, \zeta, \xi)$ [$\eta \equiv (\eta_i)$, $\xi = (\xi_{(\alpha)})$] is locally one-to-one. By the same token, the state S locally is characterized completely by the set (η, ζ, ξ) or, more conveniently, by the set (V, τ, ζ, ξ) .

We assume also that $N \neq 0$, $\tau \neq 0$. Formally this hypothesis is not necessary and is made only so as to be able to write the concluding expression in the traditional physical form. Since $\tau \neq 0$, we can put $T \equiv -1/\tau$.

From the hypotheses made, it follows that the functions ρ , P , N , $C^{(\alpha)}$, and S must satisfy the condition

$$Td(S/N) = d(c^2\rho/N) + Pd(1/N) - \sum_{\alpha=1}^{v-1} \mu_{(\alpha)}d(C^{(\alpha)}/N) \quad (3.7)$$

[$\mu_{(\alpha)}(\tau, \zeta, \xi)$ are functions (see also [15, 16])].

If the system of conservation laws (3.3)-(3.6) is complete, and we assume that the integrating factors have the tensor type indicated above ($N \neq 0$, $\tau \neq 0$), it follows that condition (3.7) is then valid. Conversely, if we have Eqs. (3.3)-(3.5), where condition (3.7) holds,

relation (3.6) is then satisfied. For this we need to multiply Eq. (3.3) by $\eta_i \equiv \tau v_i$, Eq. (3.4) by $\zeta \equiv \tau z$, and Eq. (3.5) by $\xi(\alpha) \equiv \tau \mu(\alpha)$, where $z \equiv \left(c^2 \rho + P - ST - \sum_{\alpha} \mu(\alpha) C^{(\alpha)} \right) / N$.

Thus, concepts of invariance, in combination with the requirement of completeness of the conservation laws, again rigidly determine the structure of the equations and lead to the Gibbs differential form (3.7), which is usually regarded as an expression of the second law of thermodynamics. This fact deserves attention since unexpectedly the thermodynamics here receives a new formal expression and turns out to be an internal property of a definite class of differential equations. However, one should not assume that thermodynamics must be entirely reduced to the formal principles described above and their possible generalizations. The Gibbs differential form retains its significance of a special fundamental principle, making it possible to relate two different theoretical schemes of formal and physical content, in turn. Thus, on the one hand, we can bring in a physical meaning to the formal constructions and assign to them a physical perspective; on the other hand, formalization of physical principles creates for them the perspective of new generalizations, and the possibility of applying them in new situations.

4. Principles of Thermodynamics and Differential Equations. We now examine more carefully the principles ("laws") of thermodynamics.

1. The first law of thermodynamics is a statement of the law of conservation of energy. This statement is formalized here in the law of conservation of energy (of mass), of momentum

$$\partial T^{ij} / \partial x^j = 0, \quad (4.1)$$

and, in a more general scheme, is expressed in the acceptance of a basic system of conservation laws, i.e., in the requirement of divergence of the initial set of thermodynamic equations.

2. The second law includes a complex of statements, one of which relates to the assertion of the irreversibility of processes and to dissipative phenomena. We shall not consider this circle of ideas here. Another set of ideas is connected with the Gibbs differential form and, in one of the widely prevalent expressions, with a postulate concerning existence of the integrating factor $1/T$. The integrating factor for the conservation law (4.1) turns out to be equal to the "temperature vector" $(\tau v_k) = (-1/T)v_k$ (for a Lorentzian system, the axiomatics for which are simpler and more straightforward than for a Galilean system). Of course, such a narrow correspondence is not accidental, but testifies to the reasonableness of the postulates adopted.

3. In thermodynamics there is yet a third law, introduced by Nernst and Planck. It describes the well-defined uniformity in the behavior of matter as $T \rightarrow 0$, i.e., in other words, the case of uniform degeneration in the physics of possible equations of states.

4. It is possible that the third law has, in its way, a symmetric complement at the opposite end of the temperature scale, namely a fourth law. It is entirely plausible that as $T \rightarrow \infty$ the possible forms of the equations of state may again degenerate, with the asymptotic relation,

$$P = c^2 \rho / 3 \quad (4.2)$$

(see, for example, [15]). Of course, relation (4.2) has meaning only for a Lorentzian system. Equation (4.2) is equivalent to the equation $T_j^i = 0$, where, as usual, $T_k^i \equiv g_{kj} T^{ji}$. For simplicity, let $v = 1$, i.e., the basis system consists only of Eqs. (3.3) and (3.4). The differential form (3.7) takes on the form

$$Td(S/N) = d(c^2 \rho / N) + Pd(1/N). \quad (4.3)$$

From Eqs. (4.2) and (4.3) we obtain the equation of state

$$S/N = G(\eta), \quad \eta \equiv (c^2 \rho)^3 / N^4 = (c^2 \rho / N)^3 (1/N) \quad (4.4)$$

[$G(\eta)$ is some function]. From Eqs. (4.3) and (4.4) we obtain T and P , where P coincides, of course, with relation (4.2), and

$$T = (1/3)(c^2 \rho / N) \{ 1 / (\eta \partial G / \partial \eta) \}. \quad (4.5)$$

It turns out that if an equation of state satisfies Eqs. (4.2)-(4.5), the system (3.3), (3.4) is invariant with respect to conformal transformations of the coordinates (all the conformal transformations are described in [17]). Thus, if as $T \rightarrow \infty$ the relation (4.2) holds, an additional symmetry arises asymptotically. The equations of an electromagnetic field, as is well known, are also conformally invariant. One gains the impression that as $T \rightarrow \infty$ we have, for all materials and for the field, a universal symmetry, i.e., conformal invariance. If this is true, it must then be regarded as exact and a general expression of the fourth law. In addition, this becomes an additional confirmation of our statement in the introduction to the effect that a group is the most fundamental object (along with the special divergent structure of the dynamical equation).

It would be very interesting if one could find a similar expression also for the third law, i.e., if one could find a universal (for all materials) invariance of an appropriate set of equations and/or an additional conservation law.

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